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KINEMATIC FUNCTIONS FOR THE 7 DOF ROBOTICS RESEARCH ARM

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Abstract

The Robotics Research Model K–1207 manipulator is a redundant 7R serial link arm with offsets at all joints. To uniquely determine joint angles for a given end–effector configuration, the redundancy is parameterized by a scalar variable which corresponds to the angle between the manipulator elbow plane and the vertical plane. The forward kinematic mappings from joint–space to end–effector configuration and elbow angle, and the augmented Jacobian matrix which gives end–effector and elbow angle rates as a function of joint rates, are also derived.

1. Introduction

The Robotics Research Model K–1207 arm is a seven degree–of–freedom serial link manipulator which offers one extra degree of joint–space redundancy over that needed for the fundamental task of end–effector placement and orientation. In this paper, a reasonable task–space parameterization, ψ, is first given of the redundancy, and the forward kinematic mappings from joint space to end–effector configuration and ψ are then derived. We also give the augmented Jacobian, J^A, which gives end–effector rates and $\dot{\psi}$ as a function of joint rates. A longer and more complete version of this paper is available which contains proofs, as well as an analysis of the kinematic and algorithmic singularities of the augmented Jacobian.

2. Forward Kinematics

2.1. Mapping from Joint–Space to End–Effector Configuration

The Robotics Research Model K–1207 arm is essentially a 7R spherical–revolute–spherical manipulator, but with additional nonzero offsets (denoted by the link lengths a_i, $i = 1, \cdots, 6$) at each of the joints, as shown in Figures 1–3. Denavit–Hartenberg (D–H) link frame assignments are given in accordance with the convention described in [1]. This assignment results in the following general form of the interlink homogeneous transformation matrix:

$$
{}^{i-1}T_i = \begin{pmatrix} {}^{i-1}R_i & {}^{i-1}P_i \\ 0^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & \alpha_{i-1} \\ \sin\theta_i \cdot \cos\alpha_{i-1} & \cos\theta_i \cdot \cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i \cdot \sin\alpha_{i-1} \\ \sin\theta_i \cdot \sin\alpha_{i-1} & \cos\theta_i \cdot \sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i \cdot \cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

where θ_i denotes the i^{th} joint angle. The D–H parameters for the K–1207 arm are given in Table 1. The link frame assignments for the K–1207 are given in Figure 2, where the arm is shown in its zero configuration. The link i coordinate frame is denoted by $\mathcal{F}_i$, with coordinate axes $(\widehat{x}_i, \widehat{y}_i, \widehat{z}_i)$ and origin O_i. The associated interlink transformation matrices, ${}^{i-1}T_i$, $i = 1, \cdots 7$, are easily found from the above expression evaluated for the D–H parameter values listed in Table 1. If the link length parameters a_i, $i = 1, \cdots, 6$ are set to zero, the 7R anthropomorphic arm described in [2] is retrieved and we call this arm the "zero–offset" arm.

The forward kinematic function, ${}^0T_7$, which gives the position and orientation of the end-effector as a function of joint angles $\theta = (\theta_1, \cdots, \theta_7)^T$, is ${}^0T_7 = {}^0T_1 \cdots {}^6T_7$. If these multiplications are performed to obtain a symbolic form for ${}^0T_7$, the resulting expression will be complex due to the multitude of nonzero link offsets and the fact that no two consecutive joint axes are parallel. Rather than construct and implement the symbolic expression, it is more efficient to compute the forward kinematic function ${}^0T_7$ via a link-by-link iteration of the form

$$
{}^0T_i = {}^0T_{i-1} \cdot {}^{i-1}T_i , \quad i = 1, \cdots, 7 \tag{1}
$$

thus exploiting special structural properties of the homogeneous transformation matrices during each link update. Furthermore, it is useful to explicitly have the interlink homogeneous transformations, ${}^{i-1}T_i$, since important quantities — such as the vectors w, e, and p defined later — can then be computed. In fact, such quantities are often a direct result of the intermediate steps of the iteration (1).

2.2. Mapping from Joint-Space to Elbow Angle

When the arm is in a kinematically nonsingular configuration, there will generally exist one excess joint degree-of-freedom for the task of end-effector control since there are seven joint angles available to orient and position the end-effector — a task which requires only six degrees of freedom. As a result, for a fixed end-effector configuration there is generally a one-dimensional subset of joint space (a "self-motion manifold") which maps to this configuration. Actually, there are finitely many, up to 16 in the most general case, such manifolds or "poses" [3, 4]. The extra degree of freedom represented by a self-motion manifold can be used to attain some additional task requirement, provided that this task can be performed independently of end-effector placement [5, 6]. Furthermore, the imposition of an auxiliary task constraint can provide sufficient additional information to uniquely determine the joint angles (modulo the remaining finitely many-to-one mapping property represented by the pose [3, 4]). This scalar additional task variable is denoted by ψ and is assumed to be a meaningful parameterization of the self-motion manifolds which map to a given end-effector configuration. We say that the "basic" task of end-effector placement has been *augmented* by the additional task represented by ψ. In essence, the concept of the forward kinematic map is generalized to be the (finitely many-to-one) mapping from $\theta \in R^7$ to $({}^0T_7, \psi)$.

Although ψ can be any additional scalar parameter which is independent of end-effector configuration, we define and use the "elbow angle" to resolve the manipulator redundancy. Refer to Figures 3 and 4 where $S = O_1$, $E = O_4$, and $W = O_7$ denote the origins of link frames 1, 4, and 7 respectively. ψ is defined by the angle from the vertical plane containing the shoulder-wrist line (line SW) to the shoulder-elbow-wrist plane (plane SEW) in the right-hand sense about the vector $w = W - S$. Assuming that the elbow angle ψ is a meaningful parameterization of manipulator redundancy, a self-motion is described by a rotation of the plane SEW about the line SW. Note that the elbow angle ψ is undefined when the wrist point W is anywhere on a line above the shoulder point S — even though this is generally not a singular configuration — since in this case the vertical plane is not uniquely defined. ψ is also undefined when e and w are collinear since then the plane SEW is not uniquely defined. In the latter case, the arm is either nearly fully outstretched, or folded, and is therefore near or at an "elbow singular" configuration [3, 7].

To derive the forward kinematic function which gives ψ as a function of joint angles, again consider Figure 4. Let $w = W - S$, $e = E - S$, and let $\widehat{V}$ denote the unit vector in the vertical direction of the base frame. Let the projection of e onto w be given by $d = \widehat{w}(\widehat{w}^T e)$, $\widehat{w} = w/\|w\|$. The minimum distance from the line SW to the point E is along the vector $p = e - d = (I - \widehat{w}\widehat{w}^T)e$. The vertical plane is the plane which contains both w and the vertical unit vector $\widehat{V}$. The unit vector in the vertical plane which is orthogonal to w is given by $\widehat{\ell} = \ell/\|\ell\|$, with $\ell = (w \times \widehat{V}) \times w$. We also define the unit vector $\widehat{p} = p/\|p\|$. Note that e, w, $\widehat{w}$, d, p, $\widehat{p}$, ℓ, and $\widehat{\ell}$ can be computed during the forward kinematics iteration (1) (see the discussion following equation (6) below).

The vector ℓ, or equivalently $\widehat{\ell}$, is treated as a free vector which can slide along the line SW. In particular, ℓ is moved along the line SW until its base is in contact with the base of vector p at the point d (see Figure 4), so that ψ is the angle from ℓ to p. This construction results in

$$c_\psi = \widehat{\ell}^T \widehat{p}\,, \quad s_\psi \widehat{w} = \widehat{\ell} \times \widehat{p}\,, \quad s_\psi = \widehat{w}^T (\widehat{\ell} \times \widehat{p}) \tag{2}$$

where $c_\psi = \cos\psi$ and $s_\psi = \sin\psi$. This gives

$$\tan\psi = \frac{\widehat{w}^T(\widehat{\ell} \times \widehat{p})}{\widehat{\ell}^T \widehat{p}} = \frac{\widehat{w}^T(\ell \times p)}{\ell^T p} \tag{3}$$

The result (3) can be simplified somewhat. Defining $g = w \times \widehat{v}$, we have $\ell = g \times w$, and we note that $\ell^T g = \widehat{V}^T g = 0$. This means that ℓ and $\widehat{V}$ are coplanar, both lying in the vertical plane. Since, in general, the vertical plane is spanned by $\widehat{V}$ and $\widehat{w}$, we have

$$\widehat{\ell} = \alpha \widehat{V} + \beta \widehat{w}\,, \quad \alpha = 1/(\widehat{\ell}^T \widehat{V})\,, \quad \beta = -\alpha \widehat{w}^T \widehat{V} \tag{4}$$

Substituting this result into (2) gives

$$c_\psi = \alpha \widehat{V}^T \widehat{p}\,, \quad s_\psi = \alpha \widehat{w}^T (\widehat{V} \times \widehat{p})$$

which can be used with (3) to obtain

$$\tan\psi = \frac{\widehat{w}^T(\widehat{V} \times \widehat{p})}{\widehat{V}^T \widehat{p}} = \frac{\widehat{w}^T(\widehat{V} \times p)}{\widehat{V}^T p} \tag{5}$$

Equation (5) immediately gives the forward kinematic function which maps the joint angles θ to the elbow angle ψ:

$$\psi = \operatorname{atan2}(\widehat{w}^T(\widehat{V} \times p), \widehat{V}^T p) \tag{6}$$

Note that (6) is undefined when both arguments are simultaneously zero. This occurs when the arm is in a configuration for which e and w are collinear, or for which the wrist point W is directly above the shoulder point S on the line through $\widehat{V}$. These indeterminacies are discussed above, and are due to the inability to uniquely define the elbow plane SEW or the reference vertical plane, respectively.

The augmented forward kinematics mapping $\theta \to ({}^0T_7, \psi)$ is given by (1) and (6). The quantities $\widehat{w}$ and $p = e - d = (I - \widehat{w}\widehat{w}^T)e$ are first computed during the iteration (1), after which ψ is computed by (6). Note that, with

$${}^0T_4 = \begin{pmatrix} {}^0R_4 & {}^0P_4 \\ 0^T & 1 \end{pmatrix} \quad \text{and} \quad {}^0T_7 = \begin{pmatrix} {}^0R_7 & {}^0P_7 \\ 0^T & 1 \end{pmatrix}$$

quantities which are directly computed during the iteration (1), the representations of e and w in the base (link 0) frame $\mathcal{F}_0$ are precisely ${}^0e = {}^0P_4$ and ${}^0w = {}^0P_7$. Also note that $\widehat{V}$ is a constant vector which is usually expressed in a frame which gives it a particularly simple form such as $(0,0,1)^T$ or $(1,0,0)^T$.

3. Differential Kinematics

3.1. Manipulator End–Effector Jacobian, J^{ee}

To present actual values for the end–effector Jacobian, J^{ee}, it is first necessary to choose a "velocity reference point," as well as a frame in which to represent the vectorial quantities which define the columns of the Jacobian. In this section, to simplify notation, we will suppress the trailing superscript and write the end–effector Jacobian simply as $J = J^{ee}$. When a velocity reference point, a, and a representation frame, $\mathcal{F}_r$, have been chosen (as discussed immediately below), we write ${}^rJ_a = {}^rJ_a^{ee}$.

Let ω_a and v_a be the angular and linear velocities of a coordinate frame, $\mathcal{F}_a$, located at a point a and fixed with respect to the manipulator end–effector. The point a is known as a "velocity reference point" of the end–effector. The Jacobian, $J_a(\theta) \in R^{6\times 7}$, relates joint rates to the frame $\mathcal{F}_a$ rate of change via the linear relationship $(\omega_a^T, v_a^T)^T = J_a(\theta)\dot{\theta}$ and is given by [8]

$$J_a = \begin{pmatrix} \hat{z}_1 & \cdots & \hat{z}_7 \\ \hat{z}_1 \times P_{a,1} & \cdots & \hat{z}_7 \times P_{a,7} \end{pmatrix} \tag{7}$$

In (7), $\hat{z}_i$ denotes the unit vector corresponding to the z-axis of link frame i (i.e. of $\mathcal{F}_i$) while $P_{a,i} \equiv P_{a,O_i} \equiv a - O_i$ is the vector from the origin, O_i, of link frame i to the point a. Note that $P_{i,i} = 0$.

Let $\mathcal{F}_b$ denote an alternative frame fixed with respect to the end–effector and located at the velocity reference point b. The relationship between joint rates and the rate of change of $\mathcal{F}_b$ is given by $(\omega_b^T, v_b^T)^T = J_b\dot{\theta}$. Let $\mathcal{F}_r$ and $\mathcal{F}_s$ be frames which are *not* necessarily fixed with respect to the end–effector. The representations of ω_a and v_a in frame $\mathcal{F}_r$ are denoted by ${}^r\omega_a$ and ${}^rv_a$. Similarly, ${}^s\omega_b$ and ${}^sv_b$ are the representations of ω_b and v_b in $\mathcal{F}_s$. Note that we have defined a and b to be end–effector reference points, i.e. to be fixed with respect to the end–effector, while we have placed no constraints on r and s.

The Jacobian, ${}^rJ_a$, giving the rate of change of $\mathcal{F}_a$ represented in $\mathcal{F}_r$, is related to ${}^sJ_b$, the Jacobian giving the rate of change of frame $\mathcal{F}_b$ represented in frame $\mathcal{F}_s$, by [1, 9]

$${}^rJ_a = \begin{pmatrix} {}^rR_s & 0 \\ 0 & {}^rR_s \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ {}^s\tilde{P}_{a,b} & I \end{pmatrix} \cdot {}^sJ_b \tag{8}$$

where, for a 3-vector x, $\tilde{x}$ denotes the 3×3 skew symmetric matrix defined by $\tilde{x}y = x \times y$ for every $y \in R^3$ and $P_{a,b} = a - b$. When $r = b$, we write ${}^bP_a \equiv {}^bP_{a,b}$. ${}^rR_s \in R^{3\times 3}$ is a rotation matrix, represented in frame $\mathcal{F}_r$, which gives the orientation of frame $\mathcal{F}_s$ with respect to frame $\mathcal{F}_r$. Common choices of ${}^rJ_a$ are given by ${}^7J_7$, and ${}^0J_7$. It is straightforward to show from (8) that

$$\det {}^rJ_a(\theta)\,{}^rJ_a(\theta)^T = \det {}^sJ_b(\theta)\,{}^rJ_b(\theta)^T \tag{9}$$

for every a, b, r, and s. Since an $m \times n$ matrix M, $m < n$, is full rank if and only if $\det MM^T \neq 0$, eq. (9) shows that the singularity of a manipulator Jacobian is independent of the choice of velocity reference point and representation frame, and is a function purely of the manipulator configuration variables θ.

An important aspect of the decomposition (8) is that s and b can often be chosen to make the Jacobian matrix have a particularly simple structure for the purposes of singularity analysis, efficient evaluation, and efficient inversion. For example, in [10] an algorithm for the efficient computation of ${}^0J_0$ is given. Note that J_0 does *not* give the velocity of the base frame, $\mathcal{F}_0$, as a function of joint rates — indeed, in most cases the base is assumed fixed and the base frame origin, O_0, cannot be a velocity reference point for the moving end-effector. Instead, J_0 is viewed as giving the velocity of a reference frame fixed with respect to the end-effector and instantaneously coincident with the base frame origin, O_0. The computation of

$$ {}^0J_0 = \begin{pmatrix} {}^0\hat{z}_1 & {}^0\hat{z}_2 & \cdots & {}^0\hat{z}_7 \\ {}^0\hat{z}_1 \times {}^0P_{0,1} & {}^0\hat{z}_2 \times {}^0P_{0,2} & \cdots & {}^0z_7 \times {}^0P_{0,2} \end{pmatrix} \tag{10} $$

where ${}^kP_{i,j} = {}^{O_k}P_{i,j}$ and $P_{i,j} = P_{O_i,O_j} = O_j - O_i$, naturally fits in with the forward kinematics iteration (1), since from

$$ {}^0T_i = \begin{pmatrix} {}^0R_i & {}^0P_i \\ 0^T & 1 \end{pmatrix} $$

${}^0P_i \equiv {}^0P_{i,0}$, we can obtain ${}^0P_{0,i} = -{}^0P_i$ and ${}^0\hat{z}_i \times {}^0P_{0,i}$ where ${}^0\hat{z}_i = {}^0R_ie_3$, $e_3 = (0,0,1)^T$. Having ${}^0J_0$, ${}^0J_7$ can then be found from (see (8))

$$ {}^0J_7 = \begin{pmatrix} I & 0 \\ {}^0\tilde{P}_7 & I \end{pmatrix} {}^0J_0 \tag{11} $$

The symbolic forms of ${}^0J_0$ and ${}^0J_7$ can be found from this procedure, but these expressions are complex and provide little insight.

In [9], the results in [10] are extended to show that taking $s = O_i$ and $b = O_j$ for an appropriate choice of link frames i and j can result in an expression ${}^iJ_j \equiv {}^{O_i}J_{O_j}$ which is not only efficient to compute, but which simplifies singularity analysis and (for nonredundant manipulators) inversion. In particular, to gain insight into the singularity structure of the K–1207 end-effector Jacobian (and to obtain alternative ways of constructing ${}^0J_0$ and ${}^0J_7$) we will let $b = 3$ (i.e., let the velocity reference point be the origin of link frame 3) and $s = 3$ (let the reference frame be link frame 3) in (9) to arrive at an expression for ${}^3J_3$. J_3 should be interpreted as giving the velocity of a fictitious tool frame which is instantaneously coincident with link frame 3. ${}^3J_3$ is found from eq. (7) by taking $P_{a,i} = P_{3,i} = P_{O_3,O_i} = O_3 - O_i$ and representing $\hat{z}_i$ and $P_{3,i}$ in link frame 3 to obtain ${}^3\hat{z}_i$ and ${}^3\hat{z}_i \times {}^3P_{3,i}$, ${}^3P_{3,i} = {}^{O_3}P_{O_3,O_i}$. The symbolic expression for ${}^3J_3$ found in this manner is given by

$$ {}^3J_3 = \left(\begin{matrix} -S_2C_3 & S_3 & 0 & 0 & S_4 \\ S_2S_3 & C_3 & 0 & 1 & 0 \\ C_2 & 0 & 1 & 0 & C_4 \\ d_3S_2S_3 + (a_2C_2 + a_1)S_5 & d_3C_3 & 0 & 0 & 0 \\ (d_3S_2 + a_2C_2 + a_1)C_3 & -d_3S_3 & 0 & 0 & -a_3C_4 - a_4 \\ 0 & -a_2 & 0 & a_3 & 0 \end{matrix} \right. $$
$$ \left. \begin{matrix} -C_4S_5 & C_4C_5S_6 + S_4C_6 \\ C_5 & S_5S_6 \\ S_4S_5 & C_4C_6 - S_4C_5S_6 \\ S_4(a_4C_5 + a_5) - d_5C_4C_5 & S_5[C_4(a_5C_6 - d_5s_6 + a_6) + a_4S_4S_6] \\ -S_5[a_3S_4 + d_5] & C_5[S_6(a_3S_4 + d_5) - a_6] - (a_5C_5 + a_4 + a_3C_4)C_6 \\ C_4(a_4C_5 + a_5) + C_5(d_5S_4 + a_3) & S_5[(a_4C_4S_6 + a_3S_6) + S_4(d_5S_6 - a_5C_6 - a_6)] \end{matrix} \right) \tag{12} $$

Having ${}^3J_3$, ${}^0J_0$ is found from (see eq. (8))

$$ {}^0J_0 = \begin{pmatrix} I & 0 \\ {}^0\tilde{P}_3 & I \end{pmatrix} \cdot {}^0J_3 = \begin{pmatrix} I & 0 \\ {}^0\tilde{P}_3 & I \end{pmatrix} \cdot \begin{pmatrix} {}^0R_3 & 0 \\ 0 & {}^0R_3 \end{pmatrix} \cdot {}^3J_3 \tag{13} $$

with ${}^0P_3 \equiv {}^0P_{3,0}$ given by

$$ {}^0P_3 = \begin{pmatrix} C_1(d_3S_2 + a_2C_2 + a_1) \\ S_1(d_3S_2 + a_2C_2 + a_1) \\ d_3C_2 - a_2S_2 \end{pmatrix} \tag{14} $$

and ${}^0R_3$ by

$$ {}^0R_3 = \begin{pmatrix} C_1C_2C_3 - S_1S_3 & -C_1C_2S_3 - S_1C_3 & C_1S_2 \\ C_1S_3 + S_1C_2C_3 & C_1C_3 - S_1C_2S_3 & S_1S_2 \\ -S_2C_3 & S_2S_3 & C_2 \end{pmatrix} \tag{15} $$

The relative simplicity of (12) not only enables one to efficiently compute ${}^0J_7$ via eqs. (11)–(15), but also allows one to gain insight into conditions leading to Jacobian singularity. In the special case of the zero–offset arm discussed in [2], corresponding to $a_1 = \cdots = a_6 = 0$, (12) simplifies to

$$ {}^3J_3 = \begin{pmatrix} -S_2C_3 & S_3 & 0 & 0 & S_4 & -C_4S_5 & C_4C_5S_6 + S_4C_6 \\ S_2S_3 & C_3 & 0 & 1 & 0 & C_5 & S_5S_6 \\ C_2 & 0 & 1 & 0 & C_4 & S_4S_5 & C_4C_6 - S_4C_5S_6 \\ d_3S_2S_3 & d_3C_3 & 0 & 0 & 0 & -d_5C_4C_5 & -d_5C_4S_5S_6 \\ d_3S_2C_3 & -d_3S_3 & 0 & 0 & 0 & -d_5S_5 & d_5C_5S_6 \\ 0 & 0 & 0 & 0 & 0 & d_5S_4C_5 & d_5S_4S_5S_6 \end{pmatrix} \tag{16} $$

3.2. Elbow Angle Jacobian, J^ψ, and the Augmented Jacobian, J^A

Let the relationship between the rate of change of a scalar additional task variable, ψ, and the joint rates be given by $\dot\psi = J^\psi\dot\theta$. The "augmented" Jacobian is given by

$$ J^A = \begin{pmatrix} J^{ee} \\ J^\psi \end{pmatrix} $$

where J^{ee} is the end–effector Jacobian discussed in Section 3.1. For the task of positioning and orienting the end–effector *augmented* by an additional task represented by ψ, the augmented Jacobian relates joint rates to the simultaneous rates of change of the end–effector and ψ. Given the end–effector Jacobian, J^{ee}, the augmented Jacobian J^A is obtained once J^ψ has been determined for a given task variable ψ. In this section, J^ψ is constructed for the case where ψ describes the angle between the vertical plane and the elbow plane SEW as defined in Section 2.

Before proceeding, it is necessary to define the Jacobians $\mathbf{E}$ and $\mathbf{W}$ which relate joint rates to $\dot e$ and $\dot w$ respectively via $\dot e = \mathbf{E}\dot\theta$ and $\dot w = \mathbf{W}\dot\theta$, where e and w are defined in Section 2.2. $\dot e$ is the linear velocity of the manipulator elbow point $E = O_4$, and $\dot w$ is the linear velocity of the wrist point $W = O_7$. We have

$$ \mathbf{E} = (\,\hat z_1 \times P_{4,1}\,,\;\; \hat z_2 \times P_{4,2}\,,\;\; \hat z_3 \times P_{4,3}\,,\;\; 0\,,\;\; \cdots\,,\;\; 0\,) \tag{17} $$

$$ \mathbf{W} = (\,\hat z_1 \times P_{7,1}\,,\;\; \cdots\,,\;\; \hat z_6 \times P_{7,6}\,,\;\; 0\,) \tag{18} $$

where $P_{i,j} = O_j - O_i$. Note that eqs. (17) and (18) are given in coordinate–free form and that to provide values for $\mathbf{E}$, or $\mathbf{W}$, a choice of reference frame for representing $\hat z_j$ and $P_{i,j}$ must be made. Also note that (compare eqs. (7) and (18)) $J_7^{ee} = \begin{pmatrix} \cdots \\ \mathbf{W} \end{pmatrix}$, so that any procedure for producing a value for $J_7 = J_7^{ee}$ (such as the one discussed following eq. (10)) automatically results in a value for $\mathbf{W}$. Furthermore, just as one can construct ${}^0\mathbf{W}$ from knowledge of ${}^{i-1}T_i$, $i = 1, \cdots, 7$ (say in the manner discussed after eq. (10)), one can readily compute values for $\mathbf{E}$ given the interlink homogeneous transformations ${}^{i-1}T_i$.

Recall the definitions of ℓ, $\hat{\ell}$, $\hat{V}$, p, $\hat{p}$, w, $\hat{w}$, and e given in Section 2.2. Also recall, as discussed in Section 2.2, that these quantities can all be computed from knowledge of the interlink homogeneous transformations ${}^{i-1}T_i$.

Lemma 3.1: The relationship between $\dot{\theta}$ and $\dot{\psi}$, where ψ is the elbow angle as defined in Section 2.2, is given by

$$\dot{\psi} = \frac{1}{\|p\|}(\hat{w} \times \hat{p})^T \dot{p} - \frac{1}{\|\ell\|}(\hat{w} \times \hat{\ell})^T \dot{\ell} \tag{19a}$$

$$= \frac{(\hat{w} \times \hat{p})^T}{\|p\|}\left\{\mathbf{E} - \frac{\hat{w}^T e}{\|w\|}\mathbf{W}\right\}\dot{\theta} + \frac{\hat{V}^T w}{\|\ell\|}(\hat{w} \times \hat{\ell})^T \mathbf{W}\dot{\theta} \tag{19b}$$

which results in

$$J^{\psi} = \frac{(\hat{w} \times \hat{p})^T}{\|p\|}\mathbf{E} + \left\{\frac{\hat{V}^T w}{\|\ell\|}(\hat{w} \times \hat{\ell})^T - \frac{\hat{w}^T e}{\|w\|\,\|p\|}(\hat{w} \times \hat{p})^T\right\}\mathbf{W} \tag{20}$$

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Since the elbow angle ψ is given by the angle from ℓ to p, it is natural that $\dot{\psi}$ should depend only on $\dot{\ell}$ and $\dot{p}$ as in eq. (19a). Equation (19a) says that only the components of $\dot{\ell}$ and $\dot{p}$ which result in an instantaneous motion of ℓ and p directly towards or away from each other can produce a change in the elbow angle, ψ. Based on our earlier discussions, it should be obvious that J^{ψ} can be constructed from knowledge of the interlink homogenous transformations ${}^{i-1}T_i$. Also note that J^{ψ} is independent of the reference frame chosen to represent the quantities in the right hand side of eq. (20).

4. Conclusions

In this paper the forward kinematic functions which give end–effector configuration and elbow angle as a function of joint angles for the Robotics Research Model K–1207 manipulator have been derived. Also given is the augmented Jacobian which relates joint rates to end–effector and elbow angle rates. Omitted derivations can be found in a longer and more complete version of this paper available from the authors. The fuller version of this paper also contains a detailed singularity analysis of the augmented Jacobian.

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References

[1] J.J. Craig, *Introduction to Robotics*, Addison–Wesley, Reading, Mass., 1986.

[2] J.M. Hollerbach, "Optimum Kinematic Design for a Seven Degree of Freedom Manipulator," 2^{nd} *Int. Symp. Robotics Research*, Kyoto, Japan, Aug. 20–23, 1984, pp. 215–222.

[3] J.W. Burdick, "Kinematic Analysis and Design of Redundant Robot Mechanisms," Ph.D. Thesis, Dept. of ME, Stanford, March 1988.

[4] J.W. Burdick and H. Seraji, "Characterization and Control of Self–Motions in Redundant Manipulators," *NASA Conference on Space Telerobotics*, January 1989, Pasadena.

[5] H. Seraji, "A New Approach to Global Control of Redundant Manipulators," *NASA Conference on Space Telerobotics*, Jan. 31-Feb. 2, 1989, Pasadena, California.

[6] O. Egeland, "Task-Space Tracking with Redundant Manipulators," *IEEE J. Robotics and Automation*, Vol. RA-3, 1987, pp. 471–475.

[7] C.W. Wampler, II, "Inverse Kinematics of a 7 DOF Manipulator," NATO Advanced Research Workshop on Robots with Redundancy: Design, Sensing, and Control, June 27–July 1, 1988, Lago di Garda, Italy.

[8] H. Asada and J.-J.E. Slotine, *Robot Analysis and Control*, Wiley–Interscience, New York, 1986.

[9] A. Fijany and A.K. Bejczy, "Efficient Jacobian Inversion for the Control of Simple Robot Manipulators," *1988 IEEE Int. Conf. Robotics and Automation*, Philadelphia, 1988, pp. 999–1007.

[10] D.E. Orin and W.W. Schrader, "Efficient Computation of the Jacobian for Robot Manipulators," *Int. J. Robotics Research*, Vol. 3, 1984, pp. 66–75.

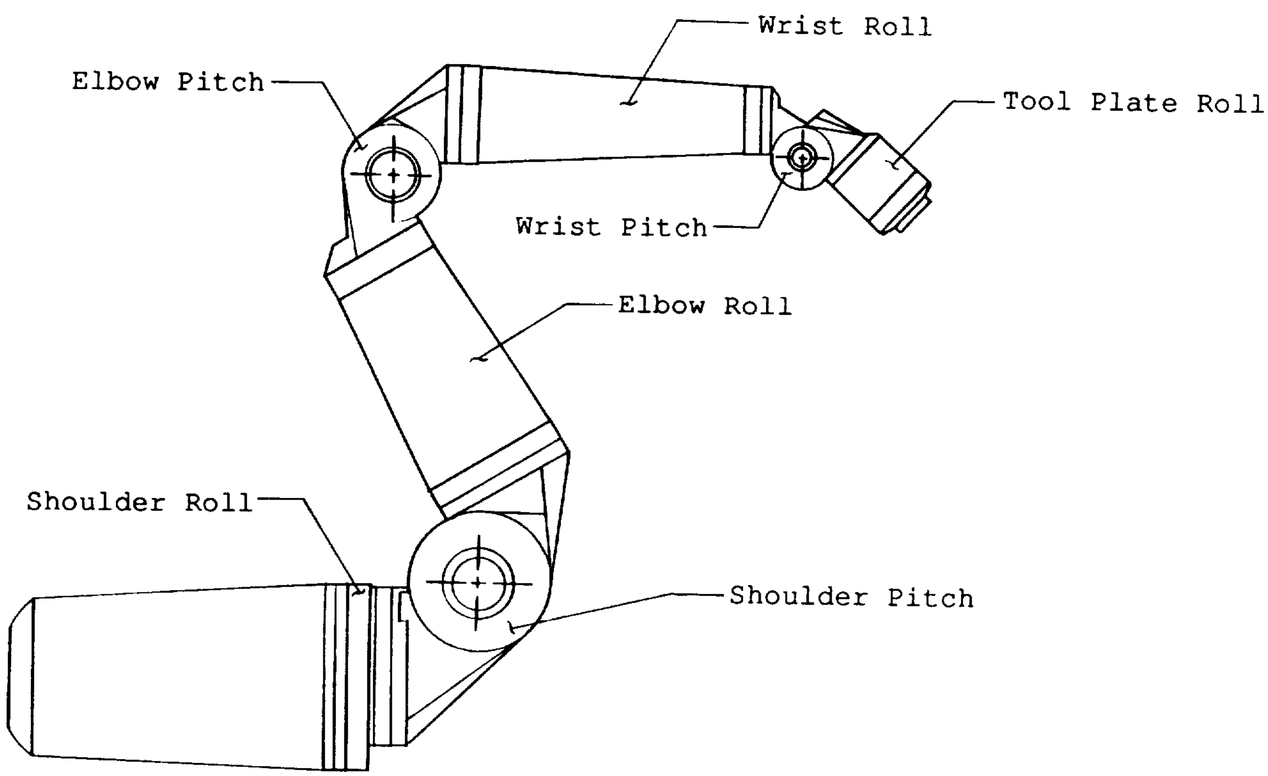

FIGURE 1: Robotics Research Model K-1207 Arm

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0°	0	0	θ_1
2	-90°	a_1	0	θ_2
3	+90°	a_2	d_3	θ_3
4	-90°	a_3	0	θ_4
5	+90°	a_4	d_5	θ_5
6	-90°	a_5	0	θ_6
7	+90°	a_6	0	θ_7

$a_1 = +4.850$ in $= +12.319$ cm
$a_2 = -4.250$ in $= -10.795$ cm
$a_3 = -3.125$ in $= -\ 7.938$ cm
$a_4 = +3.125$ in $= +\ 7.938$ cm
$a_5 = -1.937$ in $= -\ 4.920$ cm
$a_6 = +1.937$ in $= +\ 4.920$ cm
$d_3 = d_5 = 21.5$ in $= 54.61$ cm

TABLE 1: Denavit-Hartenberg Parameters of the K-1207 Arm

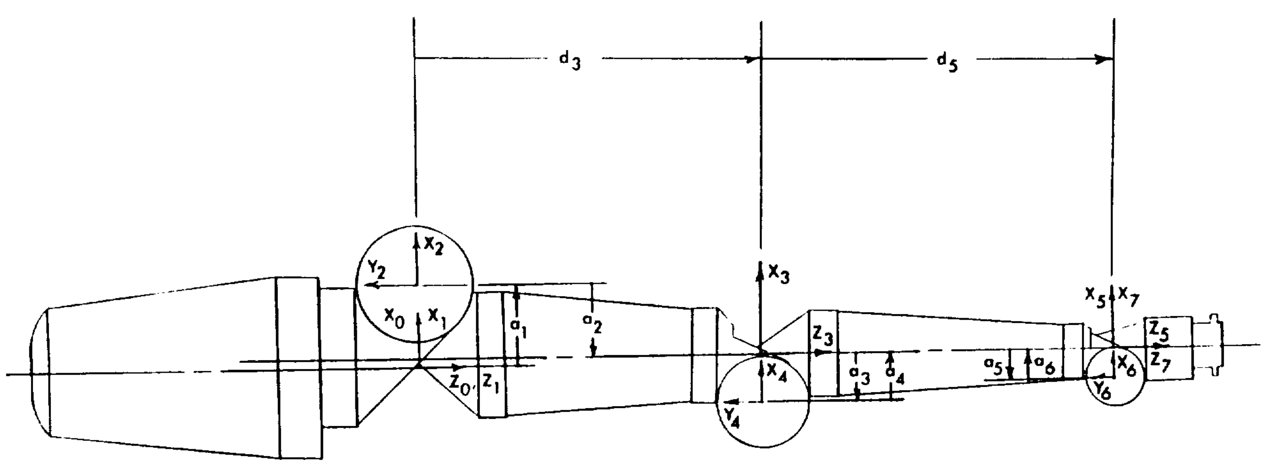

FIGURE 2: Robotics Research Model K-1207 Link Frame Assignment

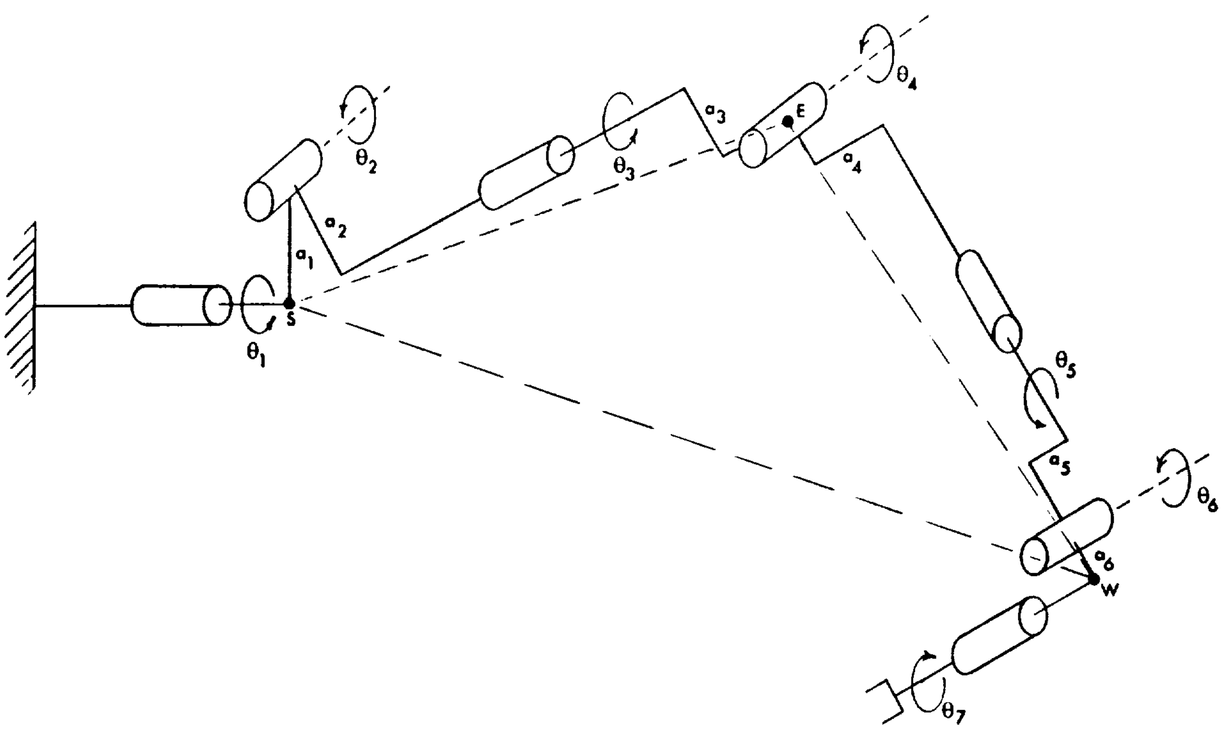

FIGURE 3: Definition of the Elbow Plane SEW

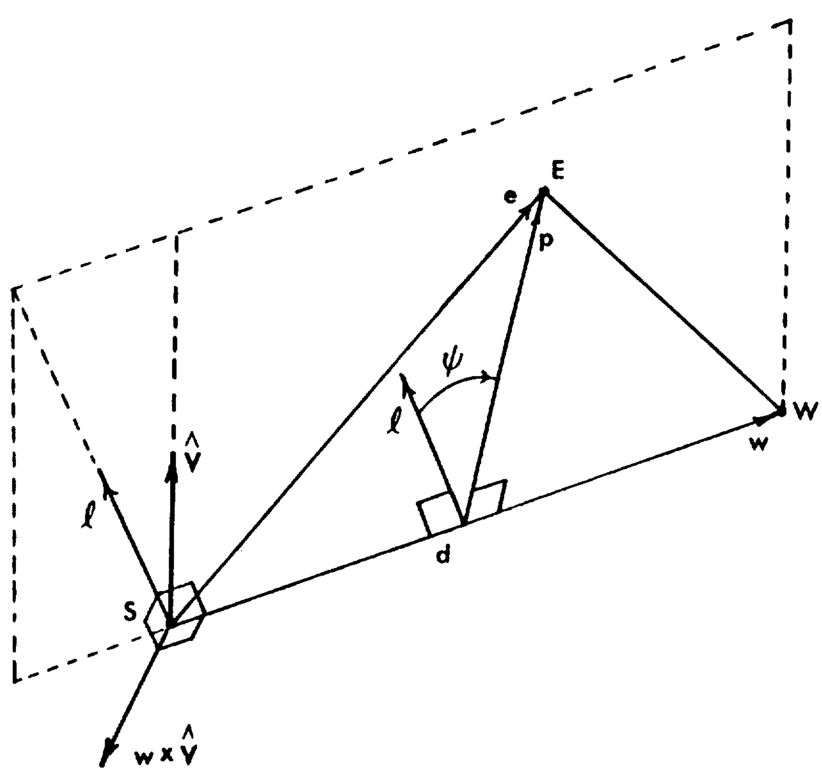

FIGURE 4: Definition of the Elbow Angle ψ